Recursive construction for a class of radial functions.
II. Superspace

Thomas Guhr\textsuperscript{a)}
Matematisk Fysik, LTH, Lunds Universitet, Box 118, 22100 Lund, Sweden
and Max Planck Institut für Kernphysik, Postfach 103980, 69029 Heidelberg, Germany

Heiner Kohler\textsuperscript{b)}
Departamento de Física, Universidad Autónoma de Madrid, Madrid, Spain
and Max Planck Institut für Kernphysik, Postfach 103980, 69029 Heidelberg, Germany

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We extend the recursion formula for matrix Bessel functions, which we obtained previously, to superspace. It is sufficient to do this for the unitary orthosymplectic supergroup. By direct computations, we show that fairly explicit results can be obtained, at least up to dimension $8 \times 8$ for the supermatrices. Since we introduce a new technique, we discuss various of its aspects in some detail. © 2002 American Institute of Physics.

I. INTRODUCTION

In a previous work, we studied properties of matrix Bessel functions in ordinary space.\textsuperscript{1} Here, we generalize these investigations to superspace. For the introductory remarks and the mathematical and physical background relevant for the ordinary space, and also relevant as the basis for the present study, we refer the reader to Ref. 1.

In mathematics, supersymmetry was pioneered by Berezin\textsuperscript{2} and, in particular group theoretical aspects, by Kac.\textsuperscript{3,4} The theory of nonlinear $\sigma$ models in spaces of supermatrix fields was developed in physics of disordered systems by Efetov.\textsuperscript{5,6} Verbaarschot, Weidenmüller, and Zirnbauer\textsuperscript{7,8} used his approach to study models in random matrix theory. In Ref. 9, the first supersymmetric generalization of the Itzykson–Zuber integral\textsuperscript{10} was given. In Ref. 11, Gelfand–Tzetlin coordinates\textsuperscript{12} were constructed for the unitary supergroup. Extending Shatashvili's\textsuperscript{13} method, the supersymmetric Itzykson–Zuber integral was also rederived in Ref. 11 in its most general form. Using the techniques of Ref. 9, such a calculation was also performed in Ref. 14.

From a mathematical viewpoint, Efetov's work\textsuperscript{5} is the basis for a harmonic analysis in certain supersymmetric coset spaces, the Efetov spaces, which are relevant for the nonlinear $\sigma$ models. In the full superspaces, a technique involving convolution integrals and ingredients of the corresponding harmonic analysis was introduced in Ref. 15. In the Efetov spaces, the theory of harmonic analysis, in both its mathematical and physical aspects, was developed by Zirnbauer\textsuperscript{16} and was applied to disordered systems in Refs. 17 and 18. In the present contribution, we do not focus on the Efetov spaces, rather we address the full supergroup spaces. The supersymmetric Itzykson–Zuber integral\textsuperscript{9} and its application in Ref. 15 is the simplest example of a supermatrix Bessel function appearing in this kind of harmonic analysis.

The matrix Bessel functions in superspace find direct application in random matrix theory. For general reviews, see Refs. 19–21. In Ref. 22 it was shown that they are the kernels for the supersymmetric analog of Dyson's Brownian motion.

The paper is organized as follows: In Sec. II, we introduce the supermatrix Bessel functions and collect basic definitions and notations. In Sec. III, we extend the recursion formula of Ref. 1 to superspaces. Since it is one of our goals to demonstrate that explicit results for supermatrix
We can now appreciate the meaning of the parameter \( c \) which enters the definition (2.3) of the real Hermitian matrices. For \( c = 1 \), it yields the real symmetric and for \( c = -1 \) the Hermitian self-dual matrix as boson–boson block, and vice versa for the fermion–fermion block. In the framework of random matrix theory, we find the supermatrices corresponding to the Gaussian orthogonal ensemble (GOE) for \( c = +1 \) and those for the Gaussian symplectic ensemble (GSE) for \( c = -1 \).

The infinitesimal volume element is given by

\[
d[\sigma] = \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} d\sigma_{ij}^{(A)} \prod_{i<j} d\sigma_{ij}^{(R)} \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} d\sigma_{ij}^{(HSd)} = \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} d\sigma_{ij}^{(A)} \prod_{i<j} d\sigma_{ij}^{(R)} \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} d\sigma_{ij}^{(HSd)},
\]
The plane waves \( \exp(i \text{trg} \sigma \rho) \) are the eigenfunctions, i.e., we have
\[
\Delta \exp(i \text{trg} \sigma \rho) = -\text{trg}^2 \exp(i \text{trg} \sigma \rho).
\]
(2.7)

Here, both \( \sigma \) and \( \rho \) are real Hermitian. As in ordinary space, the supermatrix Bessel functions are obtained by averaging over the angular coordinates, i.e., over the diagonalizing group. The Laplacian commutes with the average and we arrive at the differential equation
\[
\Delta_s \Phi_{k_1 k_2}(s,r) = -\text{trg}^2 \Phi_{k_1 k_2}(s,r),
\]
(2.8)

where the radial part of the Laplacian (2.6) reads
\[
\Delta_s = \frac{1}{\tilde{B}^{(c)}_{k_1 k_2}(s)} \left( \sum_{p=1}^{k_1} \frac{\partial}{\partial s_{p1}} \tilde{B}^{(c)}_{k_1 k_2}(s) \frac{\partial}{\partial s_{p1}} + \frac{1}{2} \sum_{p=1}^{k_2} \frac{\partial}{\partial s_{p2}} \tilde{B}^{(c)}_{k_1 k_2}(s) \frac{\partial}{\partial s_{p2}} \right).
\]
(2.9)

The Jacobian or Berezinian is given by
\[
\tilde{B}^{(c)}_{k_1 k_2}(s) = \frac{\Delta_{k_1}(s_1) \Delta_{k_2}(is_2)}{\prod_{p=1}^{k_1} \prod_{q=1}^{k_2} (s_{p1} - is_{q2})^2}, \quad \tilde{B}^{(-1)}_{k_1 k_2}(s) = \frac{\Delta_{k_1}(is_1) \Delta_{k_2}^2(s_2)}{\prod_{p=1}^{k_1} \prod_{q=1}^{k_2} (s_{p1} - s_{q2})^2}.
\]
(2.10)

One easily convinces oneself that \( \Delta_s \) depends on \( c \) only through a factor \( \sqrt{c} \). Thus, without loss of generality, we set \( c = 1 \) and omit the index \( c \).

At this point, an important comment is in order. The normalization in ordinary space according to Eq. (3.17) in Ref. 1, \( \Phi^{(c)}_N(x,0) = 1 \) and \( \Phi^{(c)}_N(0,k) = 1 \), does not carry over to the supersymmetric case. This is due to the fact that the volume of some supergroups is zero\(^2\) resulting in the vanishing of \( \Phi_{k_1 k_2}(0,s) \) for certain values of \( k_1 \) and \( k_2 \). This collides with the normalization of the plane waves (2.7) to unity at the origin. The reason for this contradiction is a well-known phenomenon in superanalysis. In going from Cartesian to angle eigenvalue coordinates, one has to add additional terms to the measure to preserve the symmetries of the original integral. These are called Efetov–Wegner–Parisi–Sourlas terms in the physics literature. A full-fledged mathematical theory of these boundary terms was given by Rothstein.\(^24\)

To solve this normalization problem, we use the following strategy. First, we evaluate the supermatrix Bessel functions without taking care of the normalization. We just multiply the integrals with a normalization constant \( \hat{G}_{k_1 k_2} \). Having done the integrals, we determine the normalization by comparing the asymptotics of the supermatrix Bessel function for large arguments with the Gaussian integral.

### III. SUPERSYMMETRIC RECURSION FORMULA

We extend the recursion formula in ordinary space\(^1\) to superspace. After stating the result in Sec. III A, we present the derivation and the calculation of the invariant measure in Secs. III B and III C, respectively.
A. Statement of the result

Let $\Phi_{k_12k_2}(s,r)$ be defined through the group integral in Eq. (2.1). It has two diagonal matrices defined as in Eq. (2.2) as arguments. It can be calculated iteratively by the recursion formula

$$\Phi_{k_12k_2}(s,r)=\hat{G}_{k_12k_2} \int d\mu(s',s) \exp(i(\text{tr}g s-\text{tr}g s')r_{11})\Phi_{(k_1-1)2k_2}(s',\bar{r}),$$  \hspace{1cm} (3.1)

where $\Phi_{(k_1-1)2k_2}(s',\bar{r})$ is the group integral (2.1) over $\text{UOSp}((k_1-1)/2k_2)$ and $\hat{G}_{k_12k_2}$ is a normalization constant, see Secs. II and VI. As in the ordinary case, the coordinates $s'$ are radial Gelfand–Tzetlin coordinates. Again, they are different from the angular Gelfand–Tzetlin coordinates which is also present in superspace, we mention that the radial measure (3.4) is quite different from the angular one.\cite{25}

The Jacobian or Berezinian consists of three parts. One of them, $\mu_B(s'_1,s_1)$, depends only on bosonic eigenvalues and one, $\mu_F(s'_2,s_2)$, on fermionic eigenvalues, i.e., only on Grassmann variables. The third part mixes commuting and anticommuting integration variables. To underline once more the difference between radial and angular Gelfand–Tzetlin coordinates which is also present in superspace, we mention that the radial measure (3.4) is quite different from the angular one.\cite{25}
As in ordinary space, the recursion formula is an exact map of the group integration onto an iteration exclusively in the radial space. Having done the iteration on the first \(k_1\) levels, we have treated all Grassmann variables. Thus, in the integrand, we are left with the matrix Bessel function \(\Phi^{(4)}_{k_2}(-i2s_2^{(k_1-1)},r_2)\) for USp(2\(k_2\)) in ordinary space.\(^1\) We notice the occurrence of the factor \(-i2\) in the argument of \(\Phi^{(4)}_{k_2}(-i2s_2^{(k_1-1)},r_2)\). Collecting everything, we arrive at

\[
\Phi_{k_1,2k_2}(s,r) = \prod_{n=1}^{k_1-1} d\mu(s^{(n)},s^{(n-1)}) \exp(i\text{trg} s^{(n-1)} - \text{trg} s^{(n)}) r_{n1} \times \exp(i\text{trg} s^{(k_1-1)} r_{k_11}) \Phi^{(4)}_{k_2}(-i2s_2^{(k_1-1)},r_2).
\]

We have set \(s = s^{(0)}\) and \(s' = s^{(1)}\). It is worthwhile to notice that the radial Gelfand–Tzetlin coordinates have a highly appreciated and valuable property: The Grassmann variables only appear as moduli squared in the integrand. Thus, the number of integrals over anticommuting variables is only half the number of the independent Grassmann variables. Moreover, advantageously, the exponential is a simple function of the integration variables. Thus, we may conclude that the radial Gelfand–Tzetlin coordinates are the natural coordinates of the matrix and the supermatrix Bessel functions, because their intrinsic features are reflected.

### B. Derivation

All crucial steps needed for the derivation of the supersymmetric recursion formula (3.1) carry over from the ordinary recursion formula in Ref. 1. We order the columns of the matrix \(u \in \text{UOSp}(k_1/2k_2)\) in the form \(u = [u_1 u_2 \cdots u_{k_1} u_{k_1+1} \cdots u_{k_1+k_2}]\). We also introduce a rectangular matrix \(b = [u_2 \cdots u_{k_1} u_{k_1+1} \cdots u_{k_1+k_2}]\) such that \(u = [u_1 b]\). Analogous to the ordinary case, we have

\[
b^\dagger b = 1_{(k_1-1)/2k_2},
\]

\[
b b^\dagger = \sum_{p=2}^{k_1} u_p u_p^\dagger + \sum_{p=k_1+1}^{k_1+k_2} u_p u_p^\dagger = 1_{2k_2} - u_1 u_1^\dagger.
\]

We define the square matrix \(\tilde{\sigma} = b^\dagger s b\) and rewrite the trace in the exponent as

\[
\text{trg } u^\dagger s u r = \text{trg } \tilde{\sigma} r + \sigma_{11} r_{11},
\]

with \(\sigma_{11} = u_1^\dagger s u_1\). Similar to the ordinary case, the first term on the right-hand side of Eq. (3.10) depends on the last \(k_1 - 1 + k_2\) columns \(u_p\) collected in \(b\) and the second term depends only on \(u_1\). Thus, it is useful to decompose the invariant measure,

\[
d\mu(u) = d\mu(b) d\mu(u_1),
\]

and to write Eq. (2.1) in the form

\[
\Phi_{k_1,2k_2}(s,r) = \int d\mu(u_1) \exp(i\sigma_{11} r_{11}) \int d\mu(b) \exp(i \text{trg } \tilde{\sigma} r).
\]

Since the coordinates \(b\) are locally orthogonal to \(u_1\), the measure \(d\mu(b)\) also depends on \(u_1\).

We now generalize the radial Gelfand–Tzetlin coordinates introduced in Ref. 1 for the ordinary spaces to the superspace. Naturally, the projector reads \((1_{k_1,2k_2} - u_1 u_1^\dagger)\) and we have the defining equation

\[
(1_{k_1,2k_2} - u_1 u_1^\dagger) s (1_{k_1,2k_2} - u_1 u_1^\dagger) e_p^\prime = s_p^\prime e_p^\prime, \quad p = 1, \ldots, k_1 - 1, k_1 + 1, \ldots, k_1 + k_2
\]
for the \((k_1-1+k_2)\) radial Gelfand–Tzetlin coordinates \(s_p'\) and the corresponding vectors \(e_p'\) as eigenvalues and eigenvectors of the matrix \((1_{k_1,2k_2}-u_1u_1^\dagger)s(1_{k_1,2k_2}-u_1u_1^\dagger)\) which has the generalized rank \(k_1-1+k_2\). Due to \(u_1^\dagger e_p'=0\), we find

\[
(1_{k_1,2k_2}-u_1u_1^\dagger)s e_p'=s_p' e_p', \quad p=1,...,k_1-1,k_1+1,...,k_2.
\]  

(3.14)

As in Ref. 11, the eigenvalues \(s_p'\) are calculated from the characteristic function

\[
z(s_p')=\det((1_{k_1,2k_2}-u_1u_1^\dagger)s-s_p')=-s_p' \detg(s-s_p')u_1^\dagger \frac{1_{k_1,2k_2}}{s-s_p'} u_1.
\]  

(3.15)

which has to be discussed in the limits

\[
z(s_p')=\begin{cases}
0 & \text{for } p=1,...,k_1-1 \\
\infty & \text{for } p=k_1+1,...,k_1+k_2.
\end{cases}
\]  

(3.16)

Thus, together with the normalization \(u_1^\dagger u_1=1\), these are \(k_1+k_2\) equations for the elements of \(u_1\).

The two parts of the integral (3.12) have to be expressed in terms of the radial Gelfand–Tzetlin coordinates \(s_p'\). In a calculation fully analogous to the ordinary case, we find

\[
\sigma_{11}=\trg s-\trg s'.
\]  

(3.17)

The eigenvalues \(t_p, p=1,...,k_1-1,k_1+1,...,k_1+k_2\) of \(\sigma\) obtain from the characteristic function

\[
w(t_p)=\detg(\sigma-t_p)=-\frac{1}{t_p} \detg((1_{k_1,2k_2}-u_1u_1^\dagger)s-t_p).
\]  

(3.18)

Comparison with Eq. (3.15) shows that the characteristic functions \(w(t_p)\) and \(z(s_p')\) are, apart from the nonzero factor \(-t_p\), identical. This implies \(t_p=s_p', p=1,...,k_1-1,k_1+1,...,k_1+k_2\).

Thus, by introducing the square matrix \(\bar{u}\) which diagonalizes \(\sigma\), we may write

\[
\bar{\sigma}=b^\dagger s b=\bar{u}^\dagger s' \bar{u}.
\]  

(3.19)

By construction, \(\bar{u}\) must be in the group \(\text{UOSp}(k_1-1/2k_2)\), because \(\sigma\) and \(\bar{\sigma}\) share the same symmetries.

These intermediate results allow us to transform Eq. (3.12) into

\[
\Phi_{k_1,2k_2}(s,r)=\int d\mu(s',s) \exp(i(\trg s-\trg s') r_{11}) \int d\mu(b) \exp(i \trg \bar{u}^\dagger s' \bar{u} \bar{r}),
\]  

(3.20)

where \(d\mu(s',s)\) is, apart from phase angles, the invariant measure \(d\mu(u_1)\), expressed in the radial Gelfand–Tzetlin coordinates \(s'\). To do the integration over \(b\), we view, for the moment, the vector \(u_1\) as fixed and observe that the measure \(d\mu(b)\) is the invariant measure of the group \(\text{UOSp}(k_1-1/2k_2)\) under the constraint that \(b\) is locally orthogonal to \(u_1\). The matrix \(\bar{u}\in\text{UOSp}(k_1-1/2k_2)\) is constructed from \(b\) under the same constraint. Thus, since \(b\) and \(\bar{u}\) cover the same manifold, the integral over \(b\) in Eq. (3.20) must yield the supermatrix Bessel function \(\Phi_{(k_1-1)2k_2}(s',\bar{r})\) and we arrive at the supersymmetric recursion formula (3.1). In the last step, we used a line of arguing slightly different from the derivation in ordinary space. In this way we avoided a discussion related to the ill-defined supergroup volume. The invariance of the measure is the crucial property we need for the proof and this holds both in superspace and in ordinary space.
C. Invariant measure

In order to evaluate the invariant measure, we have to solve the system of equations (3.15) for \(|v_p^{(1)}|^2 = |u_{p1}|^2, \ p = 1, \ldots, k_1\) and \(|\alpha_p^{(1)}|^2 = |u_{(k_1 + 2p - 1)}|^2 + |u_{(k_1 + 2p - 1)}|^2, \ p = 1, \ldots, k_2\) in terms of the bosonic eigenvalues \(s_p = s_p'\), \(p = 1, \ldots, k_1 - 1\) and the fermionic eigenvalues \(s_{k_1 + 2p} = s_{k_1 + 2p - 1} = is_p'\), \(p = 1, \ldots, k_2\),

\[
1 = \sum_{p=1}^{k_1} |v_p^{(1)}|^2 + \sum_{p=1}^{k_2} |\alpha_p^{(1)}|^2,
\]

\[
0 = \sum_{q=1}^{k_1} |v_q^{(1)}|^2 + \sum_{q=1}^{k_2} |\alpha_q^{(1)}|^2, \quad p = 1, \ldots, k_1 - 1,
\]

\[
z_p = is_p' \frac{\prod_{q=1}^{k_1} (s_{q1} - is_q') \prod_{q=1}^{k_2} (s_{q1} - is_{q2})}{\prod_{q=1}^{k_1} (s_{q1} - is_q') \prod_{q=1}^{k_2} (s_{q2} - is_{q2})}, \quad z_p \to \infty, \quad p = 1, \ldots, k_2.
\]

In Appendix A, we sketch the solution of this system for small dimensions. Inspired by these solutions one can conjecture the general solutions and verify them by plugging them directly into Eqs. (3.21)–(3.23); one finds

\[
|v_p^{(1)}|^2 = \frac{\prod_{q=1}^{k_1} (s_{p1} - is_q') \prod_{q=1}^{k_2} (s_{p1} - is_{q2})}{\prod_{q=1}^{k_1} (s_{p1} - is_q') \prod_{q=1}^{k_2} (s_{p2} - is_{q2})}, \quad p = 1, \ldots, k_1,
\]

\[
|\alpha_p^{(1)}|^2 = 2(is_p' - is_{p2}) \frac{\prod_{q=1}^{k_1} (is_{p2} - is_q') \prod_{q=1}^{k_2} (is_{p2} - is_{q2})}{\prod_{q=1}^{k_1} (is_{p2} - is_q') \prod_{q=1}^{k_2} (is_{p2} - is_{q2})}, \quad p = 1, \ldots, k_2.
\]

These expressions are reminiscent of the ones derived in Ref. 11 for unitary matrices. However, importantly, all products in (3.24) involving fermionic eigenvalues are squared. This reflects the degeneracy of \(s\) in the fermion–fermion block. We have introduced new anticommuting variables \(\xi_p'\), \(\xi_p''\) with \(|\xi_p'|^2 = is_p' - is_{p2}\) according to definition (3.7).

From this point on, the invariant measure can be calculated in the same way as for the angular Gelfand–Tzetlin coordinates, see Ref. 11 for details. The result is summarized in Eq. (3.4).

IV. THE FUNCTION \(\Phi_{22}(s,r)\)

We use the recursion formula (3.1) to calculate the supermatrix Bessel function for UOSp(2/2). To avoid the imaginary unit in the exponent, we study \(\Phi_{22}(-is,r)\). The recursion formula reads

\[
\Phi_{22}(-is,r) = \hat{G}_{22} \int d\mu(s',s) \exp((\text{tr} g - \text{tr} g')r_{11}) \Phi_{12}(-is',\bar{r}).
\]

The function \(\Phi_{12}(-is',\bar{r})\) is easily found to be

\[
\Phi_{12}(-is',\bar{r}) = \hat{G}_{12}(1 - 2(r_{21} - ir_{12})(s_{11}' - is_{12}')) \exp(2r_{12}s_{12}).
\]

The measure of the coset UOSp(2/2)/UOSp(1/2) is according to formula (3.4) given by

\[
d\mu(s',s) = \frac{(is_{12} - s_{11}') \prod_{n=1}^{2} (is_{12} - s_{1n})}{\sqrt{-\prod_{n=1}^{2} (s_{11}' - s_{1n})(is_{12} - s_{1n})^2}} ds_{11}' ds_{12}' ds_{11}. \quad d\xi_{1}'^* d\xi_{1}'.
\]
We do the Grassmann integration and find

\[ \Phi_{22}(-is,r) = \hat{G}_{22} \exp(r_{11}(s_{11} + s_{21}) - 2is_{12}r_{12}) \]

\[ \times \int_{s_{11}}^{s_{21}} \mu_B(s',s) \prod_{q=1}^{2} (is_{12} - s_{q1}) \prod_{j=1}^{2} (ir_{12} - r_{j1}) - 2(ir_{12} - r_{11}) \]

\[ \times \sum_{q=1}^{2} \frac{1}{is_{12} - s_{q1}} + 2M_{11}(s_{1}',s_{1}) \exp(s_{1}'(r_{21} - r_{11})) ds_{1}', \quad (4.4) \]

where we have introduced the operator

\[ M_{mj}(s_{1}',s_{1}) = \frac{1}{(is_{m2} - s_{rj1})} \left( \frac{1}{2} \sum_{n=1}^{k_1} \frac{1}{is_{m2} - s_{n1}} - \frac{1}{is_{m2} - s_{j1}} - \sum_{n=1}^{k_1} \frac{1}{s_{j1} - s_{n1}} - \frac{\partial}{\partial s_{j1}} \right). \quad (4.5) \]

For later purposes, we introduced general indices \( m \) and \( j \). Obviously, the Grassmann integration yielded eigenvalues in the denominator. This is somewhat surprising because of the following observation: we can always parametrize the group element \( u \in UOSp(2/2) \) in a noncanonical coset parametrization in the spirit of an Euler parametrization in ordinary space. Inserting this parametrization into the defining equation of the supermatrix Bessel function (2.1) one can expand the trace in all Grassmann variables. The expansion coefficients are polynomials in the commuting integration variables and—more important—in the matrix elements of \( s \) and \( r \). The invariant measure can be expanded in the Grassmann variables as well. It does not depend on \( r \) and \( s \). Although this procedure gets rapidly out of hand even for small groups, it is clear that the outcome of this expansion will be polynomial in the eigenvalues of \( s \) and \( r \). In other words: eigenvalues can only appear in the denominator by an integration over commuting variables and never by a Grassmann integration. Therefore, before performing any integral over commuting variables, there must exist a form of \( \Phi_{22}(-is,r) \), which is polynomial in the eigenvalues of \( s \) and \( r \).

To remove the denominators and to obtain such a polynomial expression, we use the following result. Let \( f(s_{i}') \) be an analytic, symmetric function in \( s_{i}' \), \( i = 1, \ldots, k_1 \). Furthermore, define the operator

\[ L_m(s) = \sum_{j=1}^{k_1} \frac{1}{is_{m2} - s_{j1}} \frac{\partial}{\partial s_{j1}}. \quad (4.6) \]

Then the action of the operator on the integral over the bosonic part of the measure is given by

\[ L_m(s) \int_{s_{11}}^{s_{21}} \cdots \int_{s_{(k_1-1)1}}^{s_{k_11}} \mu_B(s',s)f(s_{1}')d[s_{1}'] \]

\[ = - \int_{s_{11}}^{s_{21}} \cdots \int_{s_{(k_1-1)1}}^{s_{k_11}} \mu_B(s',s) \sum_{j=1}^{k_1-1} M_{mj}(s_{1}',s_{1})f(s_{1}')d[s_{1}']. \quad (4.7) \]

This formula is derived in Appendix B.

We now set \( f(s_{1}') = \exp(-s_{11}'(r_{21} - r_{11})) \) and insert Eq. (4.7) into Eq. (4.4), we arrive at

\[ \Phi_{22}(-is,r) = \hat{G}_{22} \exp(-2is_{12}r_{12}) \left( \prod_{j=1}^{2} (ir_{12} - r_{j1})(is_{12} - s_{j1}) \right) \]

\[ - 2 \sum_{q=1}^{2} (is_{12} - s_{q1}) \left( ir_{12} - r_{21} - r_{11} - \frac{\partial}{\partial s_{q1}} \right) \Phi_{2}^{(1)}(-is_{1},r_{1}), \quad (4.8) \]
where \( \Phi_1^{(1)}(s_1, r_1) \) is the matrix Bessel function of the orthogonal group \( O(2) \) in ordinary space as defined in Ref. 1. Although this can already be taken as the result, we underline the symmetry between \( s \) and \( r \) by using the explicit form (3.21) of Ref. 1 for \( \Phi_2(s_1, r_1) \).

\[
\Phi_2(-is, r) = \hat{G}_{22} \exp \left( \text{tr} rs - \frac{z}{2} \right) \times \left( \prod_{j=1}^{2} (is_{12} - s_{1j})(is_{12} - s_{2j}) - \frac{z}{2} \sum_{q=1}^{2} (is_{12} - s_{q1}) \right) I_0(z/2).
\]

(4.9)

where we have introduced \( z = (s_{11} - s_{21})(r_{11} - r_{21}) \) and the modified Bessel function \( I_0 \) as defined in Ref. 26.

The result (4.7) was crucial in the derivation of \( \Phi_2(-is, r) \). By means of this formula, the denominator problem was overcome in one step. Because of its importance, we want to gain more insight into this problem: In Appendix C, we rederive \( \Phi_2(-is, r) \) in two other ways. It is clear that the methods of Appendix C cannot be used for higher dimensions \( k_1 \) and \( 2k_2 \), but it will help to understand the mechanisms needed when working with radial Gelfand–Tzetlin coordinates.

V. THE SERIES OF FUNCTIONS \( \Phi_{k_14}(s, r) \)

We calculate iteratively the four supermatrix Bessel functions \( \Phi_{k_14}(s, r) \) for \( k_1 = 1, 2, 3, 4 \). We do this in Secs. VA–VD, respectively.

A. First level \( k_1 = 1 \)

According to the recursion formulas (3.1) and (3.8), the starting point is the matrix Bessel function for the unitary symplectic group \( \Phi_2^{(4)}(s_2, r_2) \), which was already calculated in Ref. 1. Up to a normalization, we have

\[
\Phi_2^{(4)}(is_2, r_2) = \sum_{\omega \in S_2} \frac{1}{\Delta_2(\omega)} \cdot \frac{1}{\Delta_2^4(\omega)} \exp(-\text{tr} is_2 \omega).
\]

(5.1)

Since the subgroup \( O(1) \) of \( \text{USp}(1/4) \) is trivial, no commuting integral has to be performed to derive \( \Phi_{14}(-is, r) \). Inserting the measure (3.4) into the recursion formula and performing the Grassmann integrations yields straightforwardly

\[
\Phi_{14}(-is, r) = \hat{G}_{14} \exp \left( \text{tr} rs \right) \left( \frac{1}{\Delta_2^2(\omega_2)} \cdot \frac{1}{\Delta_2^4(\omega_2)} \right) \exp(\text{tr} (is_2 \omega)) \times \left( \frac{2 (is_{22} - s_{11})(ir_{22} - r_{11}) - 1}{2 (is_{22} - s_{11})(ir_{22} - r_{11}) - 1} \right).
\]

(5.2)

The exchange term \( (ir_{12} - ir_{22}) \) accounts for the permutation group \( S_2 \) in Eq. (5.1). Anticipating that the structure of \( \Phi_{14}(-is, r) \) will, remarkably, survive on all levels up to \( \Phi_{44}(-is, r) \), we state that \( \Phi_{14}(-is, r) \) essentially consists of two parts. A comparison with Eqs. (4.2), and (5.1) shows that the first part of \( \Phi_{14}(-is, r) \) is a product of an exponential with three other terms. The first one,

\[
\hat{G}_{14} \cdot \frac{1}{\Delta_2^2(\omega_2)} \cdot \frac{1}{\Delta_2^4(\omega_2)},
\]

(5.3)

stems from the integral over the \( \text{USp}(4) \) subgroup. The other two terms can be identified with the supermatrix Bessel functions.
\[ \Phi_{12}(-is,r) \quad \text{with} \quad s = \text{diag}(s_{11},is_{12},is_{13}), \quad r = \text{diag}(r_{11},ir_{12},ir_{13}) \] (5.4)

\[ \Phi_{12}(-is,r) \quad \text{with} \quad s = \text{diag}(s_{11},is_{22},is_{23}), \quad r = \text{diag}(r_{11},ir_{22},ir_{23}). \] (5.5)

The second part can be considered as a correction term, which destroys the product structure of \( \Phi_{14}(-is,r) \). We may identify the different parts of the product with the integrations over the corresponding subsets of the group. Thus, \( \Phi_{22}^{(4)}(-is_2,r_2) \) arises from the integration over the USp(4) subgroup, the O(1) integration yields unity, and the other two factors come from the integration over the coset UOSp(1/4)/(USp(4) \( \otimes \) O(1)).

**B. Second level \( k_2 = 2 \)**

We now have to do one integration over a commuting variable. After the Grassmann integration, we are left with a considerable amount of terms. To arrange them in a convenient way, we introduce the following notation for the product of two operators \( D_1(s)D_2(s) \) acting on a function \( f(s) \), we define

\[ [D_1^{-1}(s)D_2(s)]f(s) = D_1(s)D_2(s)f(s) - (D_1(s)D_2(s))f(s). \] (5.6)

This means, an operator with an arrow only acts on the terms outside the squared bracket. With this notation we can write

\[ \Phi_{24}(-is,r) = \hat{G}_{24} \exp(\text{tr}(r_2s_2) + r_{21}(s_{11} + s_{21})) \int_{s_{11}}^{s_{21}} d\mu(s^\prime_1,s_1) \prod_{i=1}^{2} \prod_{j=1}^{2} (is_{i2} - s_{j1}) \]

\[ \times \left( \frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)} + \frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)} \right) \]

\[ \times \left( \frac{4}{s_{11} - s_{11}}M_{21}(s^\prime_1,s_1) - \frac{2}{s_{22} - s_{11}}M_{11}(s^\prime_1,s_1) \right) \]

\[ + \frac{2}{\Delta_2^2(is_2)} \left( 2 \text{tr}r_1 - \text{tr}ir_2 + \sum_{i=1}^{2} \frac{1}{is_{i2} - s_{11}} M_{11}(s^\prime_1,s_1) \right) \]

\[ - \frac{2}{\Delta_2^2(is_2)} \left( 2 \text{tr}r_1 - \text{tr}ir_2 + \sum_{i=1}^{2} \frac{1}{is_{i2} - s_{11}} M_{21}(s^\prime_1,s_1) \right) \]

\[ - \frac{4}{\Delta_2^2(is_2) \prod_{k=1}^{2} \frac{r_{21}-ir_{22}}{s_{k1}-is_{k2}}} \exp(s^\prime_1(r_{11} - r_{21})) + (ir_{12} - ir_{22}). \] (5.7)

As in Sec. IV, a denominator problem occurs. It becomes obvious in the product \( M_{11}(s^\prime_1,s_1)M_{21}(s^\prime_1,s_1) \). Thus, we expect an identity similar to formula (4.7). This identity should map a product of operators \( L_1(s)L_2(s) \) acting on the integral onto a product of operators...
$M_{11}(s'_1,s_1)M_{22}(s'_1,s_1)$ acting under the integral. Neither the outer operators, $L_m(s)$, nor the inner ones, $M_{mn}(s)$, commute. Hence, the desired identity must be a nontrivial one. It is given by the following result.

We have the same conditions as in formula (4.7), furthermore we define

$$[L_m(s)L_d(s)] = \sum_{n=1}^{k_1} \sum_{q=1}^{k_1} \frac{1}{(is_{m2}-s_{n1})(is_{1q2}-s_{sq1})} \frac{\partial^2}{\partial s_{n1} \partial s_{sq1}}. \quad (5.8)$$

Then the following formula holds

$$[L_m(s)L_d(s)] \int_{s_{11}}^{s_{21}} \cdots \int_{s_{1(1-1)}}^{s_{1(1-1)}} \mu_B(s',s) \frac{\partial^2}{\partial s_{11}} f(s_1') ds_1'$$

$$= \int_{s_{11}}^{s_{21}} \cdots \int_{s_{1(1-1)}}^{s_{1(1-1)}} \mu_B(s',s) \left[ \sum_{j=1}^{k_1-1} \sum_{k=1}^{k_1-1} M_{mj}(s'_1,s_1)M_{lk}(s'_1,s_1) \right.$$

$$- \frac{1}{is_{m2}-is_{1q2}} \sum_{j=1}^{k_1-1} \left( \frac{1}{is_{m2}-s_{j1}} M_{lj}(s'_1,s_1) - \frac{1}{is_{1q2}-s_{j1}} M_{mj}(s'_1,s_1) \right)$$

$$- \frac{1}{2} \sum_{k,j} (is_{m2}-s_{k1})(is_{1q2}-s_{j1})(is_{1q2}-s_{k1})(is_{m2}-s_{j1}) \left. \right] f(s_1') ds_1'. \quad (5.9)$$

The derivation is along the same lines as the one for formula (4.7), it also involves formula (4.7). With the identities (5.9) and (4.7) the denominator problem is again solved in one step. After some further manipulations we arrive at

$$\Phi_{24}(-is,r) = 2 \pi \hat{G}_{24} \exp \left( \text{trg } rs - \frac{z}{2} \right) \left[ \frac{1}{\Delta^2_2(is_2) + \Delta^2_2(is_2)} \right.$$  

$$\times \left[ \prod_{i=1}^{2} (r_i - ir_{12})(s_i - is_{12}) - \prod_{j=1}^{2} (s_j - is_{12})(r_i - ir_{12}) - z \frac{\partial^2}{\partial z^2} \right]$$

$$\times \left[ \prod_{i=1}^{2} (r_i - ir_{22})(s_i - is_{22}) - \prod_{j=1}^{2} (s_j - is_{22})(r_i - ir_{22}) - z \frac{\partial^2}{\partial z^2} \right]$$

$$- 2 \pi \hat{G}_{24} \exp \left( \text{trg } rs - \frac{z}{2} \Delta^2_2(is_2) \right)$$

$$\times \sum_{k=1}^{2} \prod_{j=1}^{2} (s_{ki} - is_{j2})(r_{ki} - ir_{j2}) I_0(z/2) - 2 \pi \hat{G}_{24} \exp \left( \text{trg } rs - \frac{z}{2} \right)$$

$$\times \frac{1}{\Delta^2_2(is_2) \Delta^2_2(is_2)} ((\text{trg } s)(\text{trg } r) - 1) z \frac{\partial}{\partial z} I_0(z/2) + (ir_{12} \rightarrow ir_{22}). \quad (5.10)$$

As in Sec. IV, we used the composite variable $z = (s_{11} - s_{12})(r_{11} - r_{12})$. A comparison with Eqs. (4.8) and (5.2) shows the similarity in the structures of $\Phi_{24}(-is,r)$ and $\Phi_{14}(-is,r)$. The former also decomposes into two parts. The first part is a product, whose factors can be assigned to the integrations over the different submanifolds of the group in the same way as in the case of $\Phi_{14}(-is,r)$. The other one can be interpreted as a correction term due to the noncommutativity of the operators $L_m$ in formula (5.9).
C. Third level $k_1=3$

This structure of $\Phi_{k_1}(is,r)$ emerging in the previous calculations is likely to also be present for arbitrary $k_1$. However, for $k_1>2$, we have so far not been able to treat the general case. Fortunately, in important physics applications, one matrix argument of the supermatrix Bessel function has an additional twofold degeneracy in the boson–boson block. In this case, it is possible to carry on the recursion up to $\Phi_{4d}(is,r)$ by extending the techniques developed for $k_1=1$ and $k_2=2$. Thus, from now on, we restrict ourselves to this case.

At first sight, one might hope to achieve some simplification by applying the projection procedure onto the degenerate matrix, because this results in a considerable simplification of the invariant measure. However, it turned out that the integrations are easier if one does the recursion procedure onto the degenerate matrix, because this results in a considerable simplification of the invariant measure. Hence, we use the measure as it stands in Eq. (3.4). We consider $\Phi_{3d}(is,r)$ in the case that

$$r_1 = \text{diag}(r_{11}, r_{21}, r_{21}).$$

(5.11)

Having performed the Grassmann integral, one can arrange the terms in a way similar to Eq. (5.7). The complete expression and further details are given in Appendix D. We then can use formula (5.9) and find after some further algebra

$$\Phi_{3d}(is,r) = 4 \exp(i r_{21} s_2)(r_{21} - ir_{12})(r_{21} - ir_{22}) \left[ \frac{1}{\Delta_2^2(2r_2)s_2} + \frac{1}{\Delta_2^2(2r_2)s_2} \right]$$

$$\times \left[ 4 \prod_{k=1}^{2} (r_{k1} - ir_{12}) \prod_{k=1}^{3} (s_{k1} - i s_{12}) \times \left( r_{11} + r_{21} - ir_{12} - \frac{\partial}{\partial s_{k1}} \right) \right]$$

$$\times \left[ 4 \prod_{k=1}^{2} (r_{k1} - ir_{22}) \prod_{k=1}^{3} (s_{k1} - i s_{22}) \times \left( r_{11} + r_{21} - ir_{22} - \frac{\partial}{\partial s_{k1}} \right) \right] - \frac{4}{\Delta_2^2(2r_2)s_2} \sum_{j=1}^{3} \prod_{k=1}^{2} (s_{j1} - is_{12})(s_{j1} - is_{22})$$

$$\times \left[ r_{11}^2 + r_{21}^2 + r_{11}r_{21} - (r_{11} + r_{21})(i r_{12} + i r_{22}) + i r_{12}r_{22}$$

$$- (r_{11} + r_{21} - ir_{12} - ir_{22}) \frac{\partial}{\partial s_{11}} - \frac{2}{\Delta_2^2(2r_2)s_2} \right]$$

$$\times \sum_{k=1}^{2} \prod_{j=1}^{2} \prod_{l=1}^{3} (s_{j1} - is_{12})(s_{j1} - is_{22}) \left( \frac{\partial}{\partial s_{k1}} \right)^2$$

$$\times \Phi^{(1)}_{3d}(is_1, r_1), \quad (ir_{12} \leftrightarrow ir_{22}),$$

(5.12)

where $\Phi^{(1)}_{3d}(is_1, r_1)$ is the matrix Bessel function of the orthogonal group O(3). We notice that the structure of $\Phi_{4d}(is,r)$ and $\Phi_{2d}(is,r)$ reappears in $\Phi_{3d}(is,r)$.

D. Fourth level $k_1=4$

In the calculation of $\Phi_{4d}(is,r)$, we again consider the case that the matrix $r$ is degenerate,

$$r_1 = \text{diag}(r_{11}, r_{11}, r_{21}, r_{21}).$$

(5.13)
The main problem is to find a convenient representation for the matrix Bessel function \( \Phi_3^{(1)}(-i s_1, r_1) \) appearing on the third level in Eq. (5.12). It turns out that the representation derived in Appendix B of Ref. 1 is very well suited to our purpose. Due to the degeneracy in \( r_1 \), the original threefold integral can be reduced to an integral over just one single variable

\[
\Phi_3^{(1)}(-i s_1', r_1) = \frac{\exp(r_{21} tr s_1')}{\sqrt{|r_{11} - r_{21}|}} \int_{-\infty}^{\infty} dt \frac{\exp(i(r_{11} - r_{21}) t)}{\prod_{i=1}^{3} \sqrt{s_{i1}' - it}}. \tag{5.14}
\]

Here, we again neglected the normalization because we want to fix it afterwards as explained previously. Similarly, \( \Phi_4^{(1)}(-i s_1, r_1) \) can be written as a double integral,

\[
\Phi_4^{(1)}(-i s_1, r_1) = \frac{\exp(r_{21} tr s_1)}{|r_{11} - r_{21}|} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 |t_1 - t_2| \frac{\exp(i(r_{11} - r_{21})(t_1 + t_2))}{\prod_{i=1}^{4} \prod_{i=1}^{2} \sqrt{s_{i1}' - it_i}}. \tag{5.15}
\]

Singlarities have to be taken care of appropriately. After inserting Eq. (5.10) into the recursion formula and performing the Grassmann integration, one can arrange the terms in a similar way as in the case of \( \Phi_3^{(1)}(-i s, r) \). At this point, we notice that formulas (4.7) and (5.9) need to be supplemented by further identities. We state the most important one in the following.

The same conditions as for formula (4.7) apply. Moreover, we define the operator

\[
\mathcal{L}_m(s) = \sum_{q=1}^{k_1} \frac{1}{i s m_2 - s q_1} \partial^2 s_{q1} + \frac{1}{2} \sum_{q' < n} \left( is m_2 - s q_1 (s q_1 - s n_1) \right) \left( \frac{\partial}{\partial s q_1} - \frac{\partial}{\partial s n_1} \right). \tag{5.16}
\]

Then we have

\[
\mathcal{L}_m(s) \int_{s_{j1}}^{s_{11}} \cdots \int_{s_{k_1}}^{s_{11}} \mu_B(s', s) d[s'_1] f(s'_1) = - \int_{s_{j1}}^{s_{11}} \cdots \int_{s_{k_1}}^{s_{11}} \sum_{j=1}^{4} \left[ M_{m j}(s_{j1}', s_1) \frac{\partial}{\partial s_{j1}} \right] f(s'_1) d[s'_1]. \tag{5.17}
\]

Again, the proof is along the same lines as the proof of formula (4.7) and the proof of formula (5.9) in Ref. 1.

Thus, there is a family of rules to transform operators symmetric in \( s_i \), \( L_m(s), \bar{L}_m(s) \) acting onto an integral into an operator acting under the integral. We need one more such transformation rule which tells us how the product \( [L_m(s) \bar{L}_l(s)] \) transforms into operators acting under the integral. This formula and further details are given in Appendix E. Collecting everything, we finally arrive at

\[
\Phi_4^{(1)}(-i s, r) = 4 \mathcal{G}_{44} \exp(-tr(r_2 s_2)) \prod_{i,j} (r_{i1} - i r_{j2}) \left[ \frac{1}{\Delta_2^*(i r_2) \Delta_2^*(i s_2)} + \frac{1}{\Delta_2^*(i r_2) \Delta_2^*(i s_2)} \right]
\]

\[
\times \left( 8 \prod_{i=1}^{2} (r_{i1} - i r_{i2}) \prod_{i,j}^{4} (s_{j1} - i s_{12}) + 4 \prod_{i=1}^{4} (s_{j1} - i s_{12}) \prod_{j \neq i}^{4} (r_{i1} + r_{i2} - i r_{j2} - \frac{\partial}{\partial s_{j1}}) \right)
\]

\[
\times \left( 8 \prod_{i=1}^{2} (r_{i1} - i r_{i2}) \prod_{i,j}^{4} (s_{j1} - i s_{22}) + 4 \prod_{i=1}^{4} (s_{j1} - i s_{22}) \prod_{j \neq i}^{4} (r_{i1} + r_{i2} - i r_{j2} - \frac{\partial}{\partial s_{j1}}) \right)
\]

\[
- \frac{16}{\Delta_2^*(i r_2) \Delta_2^*(i s_2)} \prod_{i,j}^{4} (s_{j1} - i s_{12})(s_{j1} - i s_{22}) \left( r_{i1}^2 + r_{i2}^2 + r_{i1} r_{i2} - (i r_{i2} + i r_{i2}) \right)
\]
\[ \times (r_{11} + r_{21}) + ir_{12}r_{22} - \frac{1}{2} \operatorname{tr} g \frac{\partial}{\partial s_{11}} - \frac{8}{\Delta^2_2 (ir_2) \Delta_2^4 (is_2)} \sum_{j=1}^{4} \prod_{i \neq j}^4 (s_{11} - is_{12}) \]

\[ \times \prod_{i \neq j}^4 (s_{11} - is_{22}) \left( \frac{\partial}{\partial s_{11}} - \frac{\partial}{\partial s_{j1}} \right) \Phi_{4}^{(1)} (-is_1, r_1) + (ir_{12} + ir_{22}). \]

(5.18)

We mention that in the derivation of this result we frequently used properties of the matrix Bessel functions \( \Phi_{3}^{(1)} (s_1, r_1) \) and \( \Phi_{4}^{(1)} (s_1, r_1) \) that only hold for the case that one matrix has an additional degeneracy.

VI. ASYMPTOTICS AND NORMALIZATION

The asymptotic behavior of the supermatrix Bessel functions calculated in the previous sections is a useful check which also allows us to fix the normalization constants. We find from the expressions in Eqs. (4.1), (4.8) and in Eqs. (5.2), (5.10), (5.12), and (5.18),

\[ \lim_{s \to 0} \Phi_{k_1, 2k_2} (-is, r) = 2^{k_1 - 2k_2} \tilde{G}_{k_1, 2k_2} \frac{\prod_{i=1}^{k_1} \prod_{m=1}^{k_2} (s_{1i} - is_{m2})(r_{11} - ir_{m2})}{\Delta_{k_2}^2 (is_2) \Delta_{k_2}^4 (ir_2)} \]

\[ \times \det \left[ \exp (2s_{2} r_{j2}) \right]_{i, j = 1 \cdots k_2} \lim_{r \to \infty} \Phi_{k_1}^{(1)} (-is_1, r_1). \]

(6.1)

In the degenerate case, each degenerate eigenvalue contributes according to its multiplicity. The asymptotics of the matrix Bessel functions of the orthogonal group is given by\(^{27,28}\)

\[ \lim_{r \to \infty} \Phi_{k_1}^{(1)} (-is_1 /t, r_1) = \mathcal{C}^{(k_1)} \frac{\det \left[ \exp (s_{n1} r_{1m} /t) \right]_{n, m = 1 \cdots k_1}}{\left[ \Delta_{k_1} (s_1) \Delta_{k_1} (r_1) \right]^{1/2}}, \]

(6.2)

where the constant can be found in Muirhead’s book.\(^{28}\)

\[ \mathcal{C}^{(k_1)} = \frac{\Gamma(k_1/2)}{k_1!} \pi^{k_1/2 - k_1/4}. \]

(6.3)

Thus we find

\[ \lim_{r \to 0} \Phi_{k_1, 2k_2} (-is/t, r) = 2^{k_1 - 2k_2} \frac{\Gamma(k_1 - 2k_2)}{\Gamma(k_1 - 4k_2)} \mathcal{C}^{(k_1)} \tilde{G}_{k_1, 2k_2} \]

\[ \times \frac{\det \left[ \exp (s_{n1} r_{1m} /t) \right]_{n, m = 1 \cdots k_1}}{\sqrt{B}_{k_1, k_2} (s) B_{k_1, k_2} (r)} \]

(6.4)

for the asymptotic behavior.

On the other hand, the supermatrix Bessel function relates to the kernel of Dyson’s Brownian motion in superspace.\(^{22}\) Due to the normalization of the Gaussian integral,

\[ \left( \frac{\pi}{2t} \right)^{-((k_1 - 2k_2)^2 + (k_1 - 4k_2))/4} 2^{k_1 - k_2 - k_1/2} \int d[\sigma] \exp \left( -\frac{1}{t} (\sigma - \rho) \right) = 1, \]

(6.5)

the kernel
\[ \Gamma_{k_1,k_2}(s,r,t) = \left( \frac{\pi}{2t} \right)^{-((k_1-2k_2)^2+(k_1-2k_2))/4} \frac{2^{k_2-k_1/2}}{t^{k_1/2}} \int_{\mu \in UOSp(k_1/2k_2)} \mathrm{d}\mu(u) \exp \left( -\frac{1}{t}(\sigma-\rho) \right) \]  

(6.6)
is also normalized. Since it is obviously connected with the supermatrix Bessel function by

\[ \Gamma_{k_1,k_2}(s,r,t) = \left( \frac{\pi}{2t} \right)^{-((k_1-2k_2)^2+(k_1-2k_2))/4} 2^{k_2-k_1/2} \exp \left( -\frac{1}{t}(\text{trg} s^2 + \text{trg} r^2) \right) \Phi_{k_1,k_2}(-is/t,r) \]  

(6.7)
we can fix the normalization by using the asymptotic behavior

\[ \lim_{r \to 0} \Gamma_{k_1,k_2}(s,r,t) = \left( \frac{\pi}{2} \right)^{-((k_1-2k_2)^2+(k_1-2k_2))/4} \frac{2^{k_2-k_1/2}}{k_1! k_2!} \frac{\det[\delta(s_{i_1}-r_{j_1})]}{k_1 \cdots k_2} \frac{\det[\delta(s_{i_2}-r_{j_2})]}{k_1 \cdots k_2} \frac{\sqrt{\mathcal{B}_{k_1,k_2}(s)}}{\mathcal{B}_{k_1,k_2}(r)} \]  

(6.8)
of the kernel. Comparing Eq. (6.4) with Eq. (6.8), we find

\[ \hat{\mathcal{G}}_{k_1,k_2} = \frac{2^{k_2-k_1/2} \Gamma(k_1/2)}{\pi^2 (k_1-2k_2)^2 + 5k_2^2/2 - k_1/2} \]  

(6.9)
We mention that this calculation also shows that the diffusion kernels of the one-point function and of the two-point function of Dyson’s Brownian motion, i.e., the function \( \Gamma_{(2k)}(s,r,t) \) which was denoted by \( \Gamma_{k}(s,r,t) \) in Ref. 22, indeed satisfy the proper initial condition.

**VII. APPLICATIONS**

Although we focus in this contribution on the mathematical aspects, we now briefly comment on a particular kind of application. As the reader will realize, our results derived in Sec. VI are, in some sense, more general than what we need in those applications on which we focus here. We take this as an indication that explicit results for even more complex supermatrix Bessel functions can also be obtained. The results of the previous sections yield the kernels of the supersymmetric analog of Dyson’s Brownian Motion for the GOE and the GSE in the cases \( k = 1 \) and \( k = 2 \). We do not present the physics background here. The reader interested in these applications is asked to consult Refs. 19–21 for generalities and Ref. 22, in particular Sec. 4.2, for the issue discussed here. In the present contribution, we use the same notations and conventions. We restrict ourselves to the transition toward the GOE and suppress the index \( c \). The corresponding formulas for the transition toward the GSE are derived accordingly. We treat the one- and two-point functions in Secs. VII A and VII B, respectively.

Forrester and Nagao\(^{29}\) derived expressions for generalized one-point functions of Dyson’s Brownian motion model with Poissonian initial conditions. They used an expansion of the Green function in terms of Jack polynomials. Datta and Kunz\(^{30}\) employed a supersymmetric technique to address the two-level correlation function of the Poisson GOE transition. They arrive at a finite number of ordinary and Grassmannian integrals which are still to be performed. In our approach, we also arrive at a representation of the correlation function in terms of a finite number of integrals. However, since we managed to integrate over almost all angular integrals in the previous sections, our result contains considerably less integrals, in particular, no Grassmannian ones. It has a clear structure due to the fact that, apart from two integrals, all others are eigenvalue integrals, i.e., live in the curved eigenvalue space of Dyson’s Brownian motion. Moreover, since our formulas for the kernel are valid on all scales, our result is also exact for finite level number.
A. Level density

We use the result (4.9), derived in Sec. IV, for the supermatrix Bessel function \( \Phi_{22} (-is,r) \). Using the replacement \( r \rightarrow (x+J) \) and \( s \rightarrow st \) and the relation (6.7), we obtain the diffusion kernel for the level density

\[
\Gamma_1(s,x+J,t) = (2\pi)^{-1/2} \frac{J_1}{2t} \exp \left( -\frac{1}{t} (s_{11} - x_1 - J_1)^2 - \frac{1}{t} (s_{21} - x_1 - J_1)^2 + \frac{2}{t} (is_{12} - x_1 + J_1)^2 \right) \times \left( -2 \frac{J_1}{t} \prod_{j=1}^2 (is_{12} - s_{j1}) + \sum_{q=1}^2 (is_{12} - s_{q1}) \right).
\]

(7.1)

We take the derivative with respect to the source variable \( J_1 \) and arrive at the level density

\[
\hat{R}_1(x_1,t) = \frac{1}{(2\pi)^{3/2} t} \int \exp \left( -\frac{1}{t} (s_{11} - x_1)^2 - \frac{1}{t} (s_{21} - x_1)^2 + \frac{2}{t} (is_{12} - x_1)^2 \right) \times \left( (is_{12} - s_{11}) + (is_{12} - s_{21}) \right) B_{21}(s) \int_0^1 d[\sigma(s)] ,
\]

(7.2)

where the Berezinian is given by Eq. (2.10) for \( k_1 = 2 \) and \( k_2 = 1 \). This result is exact for an arbitrary initial condition and for arbitrary \( N \). In the case of a diagonal matrix \( H^{(0)} \) as the initial condition, we have for arbitrary \( k \),

\[
Z_k^{(0)}(s) = \int d[H^{(0)}] P(H^{(0)}) \prod_{n=1}^N \frac{\Pi_{j=1}^k (is_{j2} - H^{(0)}_{nn})}{\Pi_{j=1}^k (s_{j1} + i\epsilon - H^{(0)}_{nn})} \right)^{1/2}
\]

(7.3)

and analogously for the GSE. This has to be used in Eq. (7.2) for \( k = 1 \).

In the limit \( t \rightarrow \infty \) the stationary distribution of classical Gaussian random matrix theory is recovered. This can be seen by rewriting Eq. (7.1) for the rescaled energy \( \tilde{x}_1 = x_1/t \) and the rescaled source variable \( \tilde{J}_1 = J_1/t \), see also Ref. 22. In this limit the initial condition yields unity and we arrive at an integral representation of the one-point correlation function of the pure GOE,

\[
R_1(x_1) = \frac{1}{(2\pi)^{3/2}} \Im \int \exp \left( -(s_{11} - x_1)^2 - (s_{21} - x_1)^2 + (is_{12} - x_1)^2 \right) \times \left( \frac{1}{is_{12} - s_{11}} + \frac{1}{is_{12} - s_{21}} \right) (s_{11} + i\epsilon)^{N/2} (s_{21} + i\epsilon)^{N/2} d[s]
\]

(7.4)

where the symbol \( \Im \) denotes the imaginary part. Equation. (7.4) is equivalent to the classical expressions for the one-point functions as in Mehta’s book.19

Finally, we state an integral expression for the one-point function for the case of Poissonian initial conditions, see Eq. (5.1) of Ref. 22. We have

\[
Z_k^{(0)}(s) = \left( \int dz p(z) \frac{\Pi_{j=1}^k (is_{j2} - z)}{\Pi_{j=1}^k (s_{j1} + i\epsilon - z)} \right)^N.
\]

(7.5)

Inserting this initial condition for \( k = 1 \) into Eq. (7.2) yields the level density of a transition ensemble between Poisson regularity and GOE in terms of a fourfold integral. A further analysis will be published elsewhere.
B. Two-point function

The result (5.18), derived in Sec. V, for the supermatrix Bessel function \( \Phi_{44}(-is,r) \) gives, with the replacement \( r \rightarrow (x+J) \) and \( s \rightarrow s/t \) and according to Eq. (6.7), the diffusion kernel for the two-point function

\[
\Gamma_4(s,x+J,t) = \exp \left( -\frac{1}{t} (\text{tr} s^2 + \text{tr} (x+J)^2) \right) \Phi_{44}(-2is/t,x+J). \tag{7.6}
\]

The derivative with respect to the source terms leaves us with the two-point correlation function

\[
\hat{R}_2(x_1,x_2,t) = \frac{2^6 G_{44}}{t^2} \int \left[ \sqrt{\Delta_4(x_1)} \right] \left[ \frac{1}{Z_2(0)(s)} \exp \left( -\frac{1}{t} (\text{tr} s_1^2 + 2x_1^2 + 2x_2^2 - 2(is_{12} - x_1)^2) \right) \right. \\
- 2(is_{22} - x_2)^2 \sum_{k,j} \left[ \frac{1}{(is_{12} - s_{k1})(is_{22} - s_{j1})} \left( \frac{2x_1}{t} - \frac{\partial}{\partial s_{j1}} \right) \left( \frac{2x_2}{t} - \frac{\partial}{\partial s_{k1}} \right) \right. \\
+ \left. 2(x_1 - x_2)(is_{12} - is_{22})(is_{12} - s_{k1})(is_{22} - s_{j1}) \left( \frac{2x_1}{t} - \frac{\partial}{\partial s_{j1}} \right) \left( \frac{2x_2}{t} - \frac{\partial}{\partial s_{k1}} \right) \right] \\
\left. \frac{t}{4} (x_1 - x_2)(is_{12} - s_{k1})(is_{22} - s_{j1})(is_{12} - is_{22}) \left( \frac{\partial}{\partial s_{j1}} - \frac{\partial}{\partial s_{k1}} \right) \right] \\
\left. \times \Phi_{44}^{(1)}(-2is_1/t,x) \right] \, dx_1 + (x_1 \leftrightarrow x_2). \tag{7.7}
\]

The last line indicates that the integral with \( x_1 \) and \( x_2 \) interchanged has to be added. Since the terms antisymmetric in \( x_1 \) and \( x_2 \) are antisymmetric in \( s_{12} \) and \( s_{22} \) as well this yields just a factor 2 in Eq. (7.7). The symbol 3 denotes a certain linear combination of \( \hat{R}_2(x_1,x_2,t) \) as explained in Ref. 22. The normalization constant obtained from Eq. (6.9) and is given by \( G_{44} = 2(2\pi)^{-4} \). This result is an exact expression for the two-point function of Dyson’s Brownian motion for every initial condition. Plugging in the initial condition of Eq. (7.5) for \( k=2 \), we find an integral representation of the two-point function for the transition toward the GOE. We notice that \( \Phi_{44}^{(1)}(-2is_1/t,x) \) is, according to Eq. (5.15), given as a double integral.

In the previous discussion, we referred to properties of the kernels which can be seen from the explicit formulas. In Ref. 22, only the explicit form of the kernel for \( \beta = 2 \) was available. However, some of the general properties of the kernels for \( \beta = 1 \) and \( \beta = 4 \) could be anticipated in Ref. 22 from scaling relations of the supersymmetric version of Dyson’s Brownian motion. The explicit formulas derived in the present contribution allow one to derive the integral representations (7.2) and (7.7) for the one-point and for the two-point function. Moreover, we emphasize that the kernels for the supersymmetric version of Dyson’s Brownian motion are the same on all energy scales.22 Thus, the integral representation derived here for the two-level correlation function is, apart from the initial condition, the same on the so-called unfolded scale which is relevant for physics applications. The initial condition on the unfolded scale is found along the lines given in Ref. 22.

VIII. SUMMARY AND CONCLUSION

We extended the recursion formula of Ref. 1 to superspace. Due to the group structures in superspace, we could restrict ourselves to the unitary orthosymplectic supergroup. As in the ordinary case, the recursion formula is an exact map of a group integration onto an iteration in the radial space. We used it to calculate explicit expression for certain supermatrix Bessel functions.
In ordinary space, we saw that the matrix Bessel functions are only special cases of the radial functions.\textsuperscript{1} We have not yet studied this further, but in our opinion a similar generalization is likely to also exist in superspace.

It is a major advantage of the radial Gelfand–Tzetlin coordinates in superspace that the Grassmann variables appear only as moduli squared. Thus, the number of Grassmann integrals is \textit{a priori} reduced by half. As we showed in detail, this is a highly welcome feature for explicit calculations. As a remarkable consequence of this recursive way to proceed, the structure of the supermatrix Bessel functions is only very little influenced by the matrix dimension. We also saw that the basic structures of the supermatrix Bessel function for smaller matrix dimensions survive during the iteration to higher ones. The matrix Bessel functions in ordinary space show similar features. There, the structure of the matrix Bessel functions is much more influenced by the group parameter $\beta$ than by the matrix dimension. However, as in ordinary space, it remains a challenge to find the structure of these functions for arbitrary matrix dimension.

An interesting feature occurred which sheds light on the general properties of the recursion. Total derivatives showed up in the integral over the commuting variables after having done the Grassmann integration. Since similar terms already occurred in ordinary space, they are likely to be an intrinsic property of the recursion formula. Here, we succeeded in constructing a series of operator identities to remove them. This was a crucial step for the application of the recursion formula. A deeper understanding of these identities is highly desired.

It should be emphasized that the total derivatives are no boundary terms in the sense of Rothstein. We showed in detail that such terms cannot occur because we always work in a compact space. Thus, according to a theorem due to Berezin,\textsuperscript{2} the transformation of the invariant measure to our radial Gelfand–Tzetlin coordinates cannot yield Rothstein boundary terms. However, if further integration over the eigenvalues is required, such terms can emerge.

As an application, we worked out some kernels for the supersymmetric analog of Dyson’s Brownian motion.

The radial Gelfand–Tzetlin coordinates are the natural coordinate system for the matrix Bessel functions in superspace. This parametrization represents the appropriate tool for the recursive integration of Grassmann variables. Once the particular features of this parametrization are better understood, they may allow for the evaluation of higher dimensional group integrals.

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APPENDIX A: RADIAL GELFAND–TZETLIN COORDINATES FOR THE UNITARY ORTHOSYMPLECTIC GROUP UOSp($k_1/2k_2$)

We wish to express the moduli squared of the elements of an orthogonal ($k_1/2k_2$) dimensional unit supervector in radial Gelfand–Tzetlin coordinates. To illustrate the mechanism, we start with the smallest nontrivial case, the group UOSp(2/4). We notice that there are at first sight minor, yet crucial, differences to the calculation in Ref. 11 where we also started with the smallest nontrivial case. The set of solutions of the Gelfand–Tzetlin equations (3.23) involves one bosonic and two fermionic eigenvalues. The eigenvalue equations read

\begin{align}
1 &= \sum_{p=1}^{2} (|v_p^{(1)}|^2 + |\alpha_p^{(1)}|^2), \\
0 &= \sum_{q=1}^{2} \left( \frac{|v_q^{(1)}|^2}{s_1 - s_1^{(1)}} + \frac{|\alpha_q^{(1)}|^2}{is_2 - s_1^{(1)}} \right),
\end{align}

(A1) (A2)
where the last equation has to be solved in the limit \( z_1 \to \infty \). The bosonic equation (A2) has a unique solution \( s_1^{(1)} = s_1' \). Taking \( s_1' \) as new parameter, Eqs. (A1) and (A2) can be solved, \[ |v_{p}^{(1)}|^2 = \frac{s_{p1} - s_{11}'}{s_{p1} - s_{11}} \left( 1 - \sum_{k=1}^{2} \frac{i s_{k11} - s_{11}'}{i s_{k2} - i s_{11}'} |a_{k}^{(1)}|^2 \right), \quad p = 1,2. \] (A4)

We insert these relations in Eq. (A3) and obtain
\[
z_1 = i s_2^{(1)} (s_1' - i s_2^{(1)}) \sum_{q=1}^{2} \frac{(s_{q1} - i s_{21}^{(1)})}{(i s_{q2} - i s_{21}^{(1)})^2} \left( 1 - \sum_{k=1}^{2} \frac{c_k}{i s_{k2} - i s_{21}^{(1)}} |a_{k}^{(1)}|^2 \right)^2 \]
with \( z_1 \to \infty \). Here, we have defined the commuting variables
\[
c_k = \frac{\Pi_{q=1}^{2} (i s_{k2} - s_{q1})}{i s_{k2} - s_{11}'}, \quad k = 1,2. \] (A6)

It remains to determine the set of solutions of the fermionic eigenvalue equation (A5). To this end, both sides are inverted
\[
0 = \sum_{q=1}^{2} (i s_{q2} - i s_{21}^{(1)}) \left( 1 - \sum_{k=1}^{2} \frac{c_k}{i s_{k2} - i s_{21}^{(1)}} |a_{k}^{(1)}|^2 \right)^2 + 2 \sum_{k=1}^{2} \frac{c_k}{i s_{k2} - i s_{21}^{(1)}} |a_{k}^{(1)}|^2 . \] (A7)

We can now take the square root on both sides
\[
0 = \sum_{q=1}^{2} (i s_{q2} - i s_{21}^{(1)}) \left( 1 - \sum_{k=1}^{2} \frac{c_k}{i s_{k2} - i s_{21}^{(1)}} |a_{k}^{(1)}|^2 \right)^2 + 2 \sum_{k=1}^{2} \frac{c_k}{i s_{k2} - i s_{21}^{(1)}} |a_{k}^{(1)}|^2 . \] (A8)

The most general form of the fermionic eigenvalue is
\[
is_2^{(1)} = a_0 + \sum_{k=1}^{2} a_{k} |a_{k}^{(1)}|^2 + a_{12} \prod_{k=1}^{2} |a_{k}^{(1)}|^2 . \] (A9)

After inserting this ansatz in Eq. (A8), we obtain two sets of solutions for the coefficients \( a_{10}, a_{112} \) and \( a_{ij} \) with \( i,1,2,j,2 \),
\[
is_1' = i s_2 + \left( c_1 + \frac{c_1 c_2}{i s_{12} - i s_{21}} |a_{11}^{(1)}|^2 \right), \quad \frac{|a_{11}^{(1)}|^2}{2}, \] (A10)
\[
is_2' = i s_2 + \left( c_2 + \frac{c_1 c_2}{i s_{22} - i s_{12}} |a_{11}^{(1)}|^2 \right), \quad \frac{|a_{22}^{(1)}|^2}{2}. \]

Remarkably, we have \( a_{12} = a_{21} = 0 \). This allows us to write the nilpotent part of \( i s_{k2}' \) as the modulus squared of a new anticommuting coordinate,
\[
is_{k2}' = i s_{k2} + |\xi_{k2}^{i}|^2 . \] (A11)

We solve Eq. (A10) for \( |a_{k}^{(1)}|^2 \), insert the results in Eq. (A4) and arrive at
\[ |b_1^{(1)}|^2 = \frac{(s_{p1} - s_{q1})^2}{(s_{p1} - s_{q1})^2}, \]  
\[ |a_1^{(1)}|^2 = 2(is_{p2} - is_{q2})^2, \quad p, q = 1, 2, \quad q \neq p. \]  
(A12)

The derivatives of the \( \Theta \) functions yield \( \delta \) distributions. Upon integration of the \( \delta \) distribution the two terms in the last integral cancel each other. Hence the last term vanishes identically. This completes the proof.

**APPENDIX B: DERIVATION OF FORMULA (4.7)**

The technique we use is an extension of the one developed in Appendix D of Ref. 1. First, we rewrite the integral in terms of \( \Theta \) functions. The left hand side reads

\[ L_m(s) \int \mu_B(s',s)f(s'_1) ds'_1 \prod_{k=1}^{k_1} \Theta(s_{k1} - s_{11}) \prod_{l=1}^{l_n} \Theta(s_{l1} - s_{1n}). \]  
(B1)

The integration domain is now the real axis for all variables. The action of \( L_m(s) \) on the integral yields:

\[ \int \left( \mu_B(s',s) \sum_{i=1}^{k_1} \sum_{j=1}^{k_1-1} \frac{1}{2} \frac{-1}{(is_{12} - s_{11})(is_{12} - s_{j1})} f(s'_1) \prod_{k=1}^{k_1} \Theta(s_{k1} - s_{11}) \prod_{l=1}^{l_n} \Theta(s_{l1} - s_{1n}) \right) ds'_1 
\]  
\[ + \int \mu_B(s',s) \sum_{i=1}^{k_1} \sum_{j=1}^{k_1-1} \frac{1}{2} \frac{-1}{(is_{12} - s_{11})(is_{12} - s_{j1})} f(s'_1) \prod_{k=1}^{k_1} \Theta(s_{k1} - s_{11}) \prod_{l=1}^{l_n} \Theta(s_{l1} - s_{1n}) ds'_1. \]  
(B2)

We decompose the first term in partial fractions and find

\[ \int \left( \mu_B(s',s) \sum_{i=1}^{k_1} \sum_{j=1}^{k_1-1} \frac{1}{2} \frac{-1}{(is_{12} - s_{11})(is_{12} - s_{j1})} f(s'_1) \prod_{k=1}^{k_1} \Theta(s_{k1} - s_{11}) \prod_{l=1}^{l_n} \Theta(s_{l1} - s_{1n}) ds'_1 \right) \]  
\[ \times \sum_{j=1}^{k_1-1} \frac{1}{(is_{12} - s_{j1})} \frac{-1}{\prod_{i=1}^{k_1} (s_{i1} - s_{j1})} \prod_{k=1}^{k_1} \Theta(s_{k1} - s_{11}) \prod_{l=1}^{l_n} \Theta(s_{l1} - s_{1n}) ds'_1. \]  
(B3)

An integration by parts yields

\[ \int \mu_B(s',s) \sum_{j=1}^{k_1-1} (\prod_{k=1}^{k_1} \Theta(s_{k1} - s_{11}) \prod_{l=1}^{l_n} \Theta(s_{l1} - s_{1n}) ds'_1) \]  
\[ + \int \mu_B(s',s) f(s') \times \sum_{i=1}^{k_1} \frac{1}{(is_{m2} - s_{11})} \frac{-1}{\prod_{j=1}^{k_1} (s_{j1} - s_{11})} \prod_{k=1}^{k_1} \Theta(s_{k1} - s_{11}) \prod_{l=1}^{l_n} \Theta(s_{l1} - s_{1n}) ds'_1. \]  
(B4)

The structure of Eq. (A12) indicates the form of the solutions for groups of higher order as they were stated in Eq. (3.24). They are checked by inserting them directly into the Gelfand–Tzetlin equations (3.23). The algebra needed is, although tedious, straightforward and similar to the one here.
APPENDIX C: ALTERNATIVE DERIVATIONS OF $\Phi_{22}(s, r)$

We present two different alternative derivations. We do this in some detail, because the calculations give helpful informations on the rôle played by the radial Gelfand–Tzetlin coordinates.

First, we use an angular parametrization of the coset $\text{UOSp}(2/2)/\text{UOSp}(1/2)$ by writing the first column of $u \in \text{UOSp}(2/2)$ as

$$ u_1 = \begin{bmatrix} \sqrt{1 - |\alpha|^2} \cos \vartheta \\ \sqrt{1 - |\alpha|^2} \sin \vartheta \\ \frac{1}{\sqrt{2}} \alpha \\ \frac{1}{\sqrt{2}} \alpha^* \end{bmatrix}. \quad (C1) $$

This is a canonical way to parametrize the supersphere $S^{1|2}$ that is isomorphic to the coset $\text{UOSp}(2/2)/\text{UOSp}(1/2)$, see Ref. 31. It coincides with a special choice of the angular Gelfand–Tzetlin coordinates. The invariant measure is in these coordinates simply $d\mu(u_1) = d\alpha \, d\sigma \, d\vartheta$. Thus, one directly obtains the volume $V(S^{1|2}) = 0$, see Ref. 31. In the parametrization of the measure by radial Gelfand–Tzetlin coordinates (4.3), one has to perform the Grassmann integration and to apply formula (4.7) to achieve this result.

Although we use a different coordinate system, we still take advantage of the recursion formula (3.1). To use it in the parametrization (C1), one has to solve the Gelfand–Tzetlin equations (3.21)–(3.23) for the eigenvalues. The unique solution of the bosonic equation (3.22) is

$$ s_{11}' = a_0 + \frac{\Pi_{i=1}^2 (s_{i1} - a_0)}{is_{12} - a_0} |\alpha|^2, \quad a_0 = \frac{s_{11} + s_{21}}{2} - \frac{s_{11} - s_{12}}{2} \cos 2 \vartheta. \quad (C2) $$

The fermionic equation yields

$$ is_{12}' = is_{12} + \frac{\Pi_{i=1}^2 (s_{i1} - is_{12})}{is_{12} - a_0} |\alpha|^2. \quad (C3) $$

After inserting Eqs. (C2) and (C3) and the measure $d\mu(u_1)$ into the recursion formula (3.1), the Grassmann integration can be performed. Remarkably, we arrive at the denominator–free expression

$$ \Phi_{22}(s, r) = \bar{G}_{22} \int_0^{2\pi} d\vartheta \exp \left( \text{tr} (rs - \frac{z}{2} + \frac{z}{2} \cos 2 \vartheta) \right) $$

$$ \times \left[ \left( \prod_{i=1}^2 (r_{i1} - ir_{12})(s_{i1} - is_{12}) + \frac{1}{2} \sum_{j=1}^2 (s_{j1} - is_{12})(r_{j1} - ir_{12}) \right) $$

$$ - \frac{1}{2} \left( ir_{12} - \frac{1}{2} (r_{11} + r_{21}) \right) (s_{11} - s_{21}) \cos 2 \vartheta - \frac{z}{8} (ir_{12} - r_{21})(s_{11} - s_{21}) \sin^2 2 \vartheta \right]. \quad (C4) $$

To make contact with Eq. (4.9) one has to realize that in Eq. (C4) an additional total derivative appears in the integrand. This becomes obvious if one adds and subtracts $z/4 \cos 2 \vartheta$ in the square bracket of Eq. (C4).
\[ \Phi_{22}(s,r) = \hat{G}_{22} \int_0^{2\pi} d\vartheta \exp \left( \text{tr} \ G - \frac{r^2}{2} + \frac{z^2}{2} \cos 2\vartheta \right) \]
\[ \times \left( \prod_{i=1}^{2} \ (r_{i1} - ir_{12})(s_{i1} - is_{12}) + \frac{1}{2} \sum_{j=1}^{2} (s_{j1} - is_{12})(r_{j1} - ir_{12}) - \frac{z}{4} \cos 2\vartheta \right) \]
\[ + \frac{ir_{12} - r_{21}}{r_{21} - r_{11}} \int_0^{2\pi} d\vartheta \frac{\partial^2}{\partial(2\vartheta)^2} \exp \left( \text{tr} \ G - \frac{z}{2} (1 - \cos 2\vartheta) \right). \] (C5)

While the first integral reproduces Eq. (4.9), the second one vanishes identically. In general, in performing Grassmann integrations, one has to take care of boundary contributions.\(^2\)\(^,\)\(^3\)\(^4\) These contributions can appear whenever even coordinates are shifted by nilpotents and the function one integrates does not have compact support.\(^2\) However, in our case the basis space is always given by an \(n\) dimensional sphere, i.e., by a compact manifold without boundary. Thus in a properly chosen coordinate system, no boundary terms should appear. With regard to Eq. (C5) this means: the fact that the last term vanishes is a direct consequence of the compactness of the circle and of the analyticity of the function, that we integrate. However, in the radial Gelfand–Tzetlin coordinates, only the moduli squared of the vector \(u_1\) are determined. Therefore, not the whole sphere, but only a \((2n+1)\)th segment of it is covered by Eq. (3.24). In our case, not the circle but only a quarter of it is parametrized. This is allowed since the supermatrix Bessel functions depend only on the moduli squared \(|u_1|^2\). Nevertheless, one has to ensure that the introduction of these artificial boundaries does not alter the result. To this end we use the following integration formula.

Let \(s_{11} < s_{11}' < s_{21}\) be real and let \(\xi', \xi'^*\) be anticommuting. Furthermore, define
\[ f(s_{11}', \xi', \xi'^*) = f_0(s_{11}') + f_1(s_{11}')|\xi|^2, \] (C6)
with two analytic functions \(f_0(s_{11}'), f_1(s_{11}')\). Then the integral
\[ I = \int_{s_{11}}^{s_{21}} ds_{11}' d\xi'^* d\xi f(s_{11}', \xi, \xi'^*) \] (C7)
transforms under a shift of \(s_{11}'\) by nilpotents
\[ s_{11}' \to y + g(y)|\xi|^2 \] (C8)
in the following way:
\[ I = \int_{s_{11}}^{s_{21}} dy d\xi'^* d\xi \frac{\partial s_{11}'}{\partial y} f(y(s_{11}'), \xi, \xi'^*) - [f_0(s_{21})g(s_{21}) - f_0(s_{11})g(s_{11})]. \] (C9)

The proof is by direct calculation. The second term in Eq. (C9) is often referred to as boundary term. It can be viewed as the integral of a total derivative, i.e., an exact one-form, that has to be added to the integration measure for functions with noncompact support.\(^2\)\(^,\)\(^3\)\(^4\) For functions of an arbitrary number of commuting and anticommuting arguments, a similar integral formula holds with additional boundary terms.\(^2\) In going from the canonical coordinates \((\partial, a, a'^*\)) to the radial ones \((s_{11}, \xi_1', \xi_1'^*)\), in principle boundary terms can arise, since the bosonic Gelfand–Tzetlin eigenvalue (C2) contains nilpotents. However, the crucial quantity is \(g(y)\) in formula (C9) which, in our case, is given by
\[ g(s_{11}') = \frac{\Pi_{i=1}^{2}(s_{i1} - s_{11}')}{{i}s_{12} - s_{11}'} . \] (C10)
Thus, \( g(s_1') \) causes the boundary term to vanish at \( s_{11} \) and \( s_{21} \). It is the product structure of the left-hand side of Eq. (3.24) which always guarantees the vanishing of the boundary terms, when one goes from the Cartesian set of coordinates to the radial Gelfand–Tzetlin coordinates.

Therefore, one may think of the denominators, arising in Eqs. (4.4) and (4.5), as belonging to total derivatives of functions, which vanish at the boundaries. Keeping this in mind we derive Eq. (4.9) in yet another way. We expand the product

\[
\prod_{q=1}^{k_1} (is_{m2} - s_{q1}) = \sum_{n=0}^{k_1} \frac{1}{n!} (is_{m2} - s_{j1}')^n \frac{\partial^n}{\partial(s_{j1})^n} \prod_{q=1}^{k_1} (s_{j1}' - s_{q1}),
\]

and insert it into the integral

\[
\int_{s_{11}}^{s_{21}} \ldots \int_{s_{(k-1)1}}^{s_{k1}} \mu_g(s_1', s_1) K_m(s_1', s_1) f(s_1') d[s_1']
\]

\[
= \int_{s_{11}}^{s_{21}} \ldots \int_{s_{(k-1)1}}^{s_{k1}} \mu_g(s_1', s_1) \prod_{n=1}^{k_1} (is_{m2} - s_{n1}) M_m(s_1', s_1) f(s_1') d[s_1']. \tag{C12}
\]

We can remove the term proportional to \((is_{m2} - s_1')^{-2}\) in the integrand by an integration by parts. Through the expansion (C11), the vanishing of the boundary terms is assured. We arrive at

\[
K_{m1}(s_1', s_1) = \frac{k_1}{2} \sum_{q=1}^{k_1} \frac{1}{is_{m2} - s_{q1}} - \frac{1}{2} \sum_{q=1}^{k_1} \frac{1}{s_{j1}' - s_{q1}} - \frac{k_1}{2} \sum_{q=1}^{k_1} \frac{1}{s_{j1}' - s_{q1}} - \frac{\partial}{\partial s_{q1}^{'}}.
\]

We notice that in the new operator \( K_{m1}(s_1', s_1) \) all denominators of the type \((is_{m2} - s_1')^{-1}\) have disappeared. For \( k_1 = 2 \), we calculate

\[
K_{11} = - (is_{12} + s_{11}' - s_{11} - s_{21}) \frac{\partial}{\partial s_{q1}^{'}}, \tag{C14}
\]

which can be inserted into Eq. (4.4) by using the definition (C12). Finally, the result (4.9) is reproduced by the substitution

\[
s_1' = \frac{s_{11} + s_{21}}{2} - \frac{s_{11} - s_{12} \cos 2 \theta}{2}, \tag{C15}
\]

see Eq. (C2). In other words, we have seen that the result of this procedure is summarized in formula (4.7).

Finally, some remarks are in order: First, from this discussion, one might conclude that the radial Gelfand–Tzetlin coordinates are less adapted to the problem than the canonical parametrization (C1), because, in the latter, no denominators appear. We stress that this is not true. Certainly, the denominators appear due to the shift of the bosonic variable by nilpotents in Eq. (C2). However, the difficulty in deriving Eq. (4.9) is the identification of the different parts of the integrand after the Grassmann integration. Some of them belong to total derivatives and this problem exists in both parametrizations. Second, we emphasize that the appearance of total derivatives in the integrand is not a peculiarity of supersymmetry. Already in Ref. 1 where the matrix
Bessel functions in ordinary space were treated we had to solve a similar problem. The appearance of these total derivatives is an intrinsic property of the recursion formula. A geometrical interpretation of this phenomenon is highly desired.

**APPENDIX D: DETAILS FOR THE DERIVATION OF $\Phi_{34}(-is,r)$**

We always consider the case that one matrix has an additional degeneracy according to Eqs. (5.11) and (5.13). We introduce the notation

$$S_{ij} = (s_{ij} - is_{ji}), \quad R_{ij} = (r_{ij} - ir_{ji}).$$

Due to the degeneracy, $\Phi_{24}(-is,\bar{r})$ simplifies enormously as compared to the general result (5.10). We insert it into the recursion formula (3.1) and do the trivial integral over the O(2) subgroup. After performing the Grassmann integrals we arrive at an expression similar to Eq. (5.7),

$$\Phi_{34}(-is,r) = 4\hat{G}_{34} \exp(\text{tr} r_2 s_2 + r_{11}(s_{11} + s_{21})) \int \frac{1}{\Delta^2_1(ir_2)\Delta^2_2(is_2)} \left( \prod_{i=1}^{2} R_1, \prod_{j=1}^{3} S_{ji} \right)$$

$$\times \left[ \frac{1}{\Delta^2_1(ir_2)\Delta^2_2(is_1)} + \frac{1}{\Delta^2_2(ir_2)\Delta^2_1(is_2)} \right] \left( \prod_{i=1}^{2} R_1 - 2 \sum_{k=1}^{3} R_{2i} S_{ki}^{-1} \right)$$

$$+ 2 \sum_{j=1}^{2} M_{ij}(s_{j1}^{'}, s_{11}) \left( \prod_{i=1}^{2} R_{1i} - 2 \sum_{k=1}^{3} R_{2j} S_{kj}^{-1} + 2 \sum_{j=1}^{2} M_{2j}(s_{j1}^{'}, s_{11}) \right)$$

$$+ \frac{1}{\Delta^2_2(ir_2)\Delta^2_1(is_2)} + \frac{1}{\Delta^2_1(ir_2)\Delta^2_2(is_2)} \right]$$

$$\times \left[ \frac{1}{\Delta^2_1(is_1 - s_{j1}^{'})} - \frac{4}{is_{22} - s_{j1}^{'}} M_{2j}(s_{j1}^{'}, s_{11}) \right]$$

$$\times \left( \text{tr} r + r_{11} - \sum_{i=1}^{2} S_{i1}^{-1} \right) \left( \prod_{j=1}^{2} M_{1j}(s_{j1}^{'}, s_{11}) - \frac{2}{\Delta^2_2(ir_2)\Delta^2_2(is_2)} \right)$$

$$\times \left( \text{tr} r + r_{11} - \sum_{i=1}^{2} S_{1i}^{-1} \right) \left( \prod_{j=1}^{2} M_{2j}(s_{j1}^{'}, s_{11}) \right)$$

$$- \frac{4}{\Delta^2_2(ir_2)\Delta^2_2(is_2)} \sum_{k=1}^{3} R_{2j} S_{kj}^{-1} \right] \left[ \exp((s_{21}^{'}, s_{11}^{'})(r_{21} - r_{11}))(i r_{12} - i r_{22}) \right].$$

(D2)

Formulas (5.9) and (4.7) are needed to remove the denominators, in a way similar as for $\Phi_{24}(s,r)$. A single sum $\sum_{j=1}^{2} M_{ij}(s_{j1}^{'}, s_{11})$ transforms according to formula (4.7). Moreover, we observe that parts of Eq. (D2) together with the product $\sum_{j=1}^{2} M_{ij}(s_{j1}^{'}, s_{11}) \sum_{k=1}^{3} R_{2j} S_{kj}^{-1}$ yield exactly the integrand of formula (5.9). Thus, it can be transformed accordingly. After rearranging terms, we arrive at the result (5.12).
**APPENDIX E: DETAILS FOR THE DERIVATION OF** $\Phi_{44}(-is,r)$

For the recursion, we need $\Phi_{34}(s,r)$ with degenerate $\vec{r} = \text{diag}(r_{11}, r_{21}, r_{21})$ according to Eq. (5.11). Using the integral representation (5.14) for $\Phi^{(1)}_3(-is', \vec{r}_1)$ we find the helpful identity

$$\frac{\partial}{\partial s'_{i1}} \frac{\partial}{\partial s_{j1}} \exp(-r_{21} \text{tr} s'_i) \Phi^{(1)}_3(-is'_i, \vec{r}_1) = \frac{1}{2} \frac{\partial}{\partial s'_{i1}} - \frac{\partial}{\partial s_{j1}} \exp(-r_{21} \text{tr} s'_i) \Phi^{(1)}_3(-is'_i, \vec{r}_1).$$

(E1)

We stress that this relation, which is crucial in the derivation, only holds, because of the degeneracy in the matrix $\vec{r}_1$. Employing Eq. (5.14) and another identity,

$$\sum_{i=1}^{3} \frac{\partial}{\partial s_{j1}} \exp(-r_{21} \text{tr} s'_i) \Phi^{(1)}_3(-is'_i, \vec{r}_1) = (r_{11} - r_{21}) \exp(-r_{21} \text{tr} s'_i) \Phi^{(1)}_3(-is'_i, \vec{r}_1).$$

(E2)

We insert $\Phi_{34}(s', \vec{r})$ into the recursion formula (3.1). We then can arrange the terms emerging from the Grassmann integration in a way similar to the former cases. We obtain

$$\Phi_{44}(-is,r) = 4 \hat{G}_{44} \exp(\text{tr} r_2 s_2 + r_{11} \text{tr} s_1) \int d\mu_p(s'_1, s_1) \prod_{j=1}^{2} R_{2j} \prod_{j=1}^{4} S_{jj}$$

$$\times \left[ \frac{1}{\Delta_2'(i r_2) \Delta_2'(i s_2)} + \frac{1}{\Delta_2'(i r_2) \Delta_2'(i s_2)} \right]$$

$$\times \left[ 8 R_{21} R_{11} - 4 R_{11} R_{21} \sum_{k=1}^{4} S_{k1}^{-1} + 4 \sum_{j=1}^{3} \left( R_{21} \frac{\partial}{\partial s_{j1}} M_{1j}(s'_1, s_1) \right) \right]$$

$$\times \left[ 8 R_{22} R_{12} - 4 R_{12} R_{22} \sum_{k=1}^{4} S_{k2}^{-1} + 4 \sum_{j=1}^{3} \left( R_{22} \frac{\partial}{\partial s_{j1}} M_{2j}(s'_1, s_1) \right) \right]$$

$$+ \frac{16}{\Delta_2'(i r_2) \Delta_2'(i s_2)} \sum_{j=1}^{3} M_{1j}(s'_1, s_1) \left( \frac{1}{2} R_{11} R_{12} \left( \text{tr} r - \sum_{i=1}^{4} S_{i1}^{-1} \right) \right)$$

$$+ \left( r_{21} - r_{11} - \frac{\partial}{\partial s_{j1}} \right) \left( R_{22} R_{12} + R_{11} R_{12} + R_{11} R_{21} + \frac{1}{2} (R_{12} + R_{22}) \sum_{i=1}^{4} S_{i1}^{-1} \right)$$

$$- \frac{16}{\Delta_2'(i r_2) \Delta_2'(i s_2)} \sum_{j=1}^{3} M_{2j}(s'_1, s_1) \left( \frac{1}{2} R_{11} R_{12} \left( \text{tr} r - \sum_{i=1}^{4} S_{i2}^{-1} \right) \right)$$

$$+ \left( r_{21} - r_{11} - \frac{\partial}{\partial s_{j1}} \right) \left( R_{11} R_{21} + R_{11} R_{12} + R_{12} R_{22} + \frac{1}{2} (R_{12} + R_{22}) \sum_{i=1}^{4} S_{i2}^{-1} \right)$$

$$- \frac{16}{\Delta_2'(i r_2) \Delta_2'(i s_2)} \sum_{k=1}^{4} \prod_{j=1}^{2} R_{ij} S_{kj}^{-1} \exp(-r_{11} \text{tr} s'_i) \Phi^{(1)}_3(-is'_i, \vec{r}_1) + C(s,r)$$

$$+ (i r_{12} \rightarrow i r_{22}).$$

(E3)

Again, all operators with an arrow are understood to act only onto the term outside the squared bracket, i.e., onto $\exp(-r_{11} \text{tr} s'_i) \Phi^{(1)}_3(-is'_i, \vec{r}_1)$. In the function $C(s,r)$, we summarized the terms that are expected to arise due to noncommutativity of some operators acting on the integral and some operators acting under the integral. The last two lines in formula (5.9) are examples of such terms.
In order to evaluate Eqs. (E3) and (E4) we need some more properties of the matrix Bessel function \( \Phi^{(1)}_{ij}(-is_1,r_1) \). We investigate the action of \( \tilde{L}_k \) on \( \Phi^{(1)}_{ij}(-is_1,r_1) \) using the integral representation (5.15).

After a straightforward calculation involving an integration by parts we find

\[
\tilde{L}_k \exp(-r_1 \text{tr } s_1) \Phi^{(1)}_{ij}(-is_1,r_1) = \sum_{i=1}^{4} \frac{1}{is_{k2} - s_{i1}} \left( (r_{21} - r_{11})^2 + (r_{21} - r_{11}) \frac{\partial}{\partial s_{i1}} \right) \exp(-r_1 \text{tr } s_1) \Phi^{(1)}_{ij}(-is_1,r_1). \tag{E5}
\]

Now Eqs. (E3) and (E4) can be enormously simplified by the observation that

\[
((r_{21} - r_{11}) L_k - \tilde{L}_k) \exp(-r_1 \text{tr } s_1) \Phi^{(1)}_{ij}(-is_1,r_1) = 0, \tag{E6}
\]

which follows directly from Eq. (E5). We find for Eq. (E3)
\[ \Phi_{4d}(-is,r) = 4\hat{G}_{44}\exp(\text{tr} \, r_2 s_2 + r_{11} \text{tr} \, s_1) \int d\mu_B(s'_1, s_1) \prod_{i=1}^{2} R_{i1} \prod_{j=1}^{4} S_{ji} \]

\[ \times \left[ \left( \frac{1}{\Delta^2_2(\text{ir}_2) \Delta^2_2(\text{i}s_2)} + \frac{1}{\Delta^2_2(\text{ir}_2) \Delta^2_2(\text{i}s_2)} \right) \left( R_{12} R_{11} \left( 8R_{12} R_{22} - 4R_{22} \sum_{k=1}^{4} S_{k1}^{-1} \right) \right) \right. 

\[ \times \left( 8R_{12} R_{21} - 4R_{21} \sum_{k=1}^{4} S_{k1}^{-1} \right) 

\[ \left. + R_{12} R_{11} \left( 8R_{12} R_{21} - 4R_{21} \sum_{k=1}^{4} S_{k1}^{-1} \right) \right) \right] 

\[ \times \exp(-r_{11}\text{tr} \, s'_1) \Phi^{(1)}_{3}(-is'_1, \bar{r}_1) + C(s, r) + (ir_{12} \rightarrow ir_{22}). \]  

(E7)

The terms contained in Eq. (E4) simplify, too. We arrive at

\[ C(s, r) = 4\hat{G}_{44}\exp(\text{tr} \, r_2 s_2 + r_{11} \text{tr} \, s_1) \int d\mu_B(s'_1, s_1) \prod_{i=1}^{2} R_{i1} \prod_{j=1}^{4} S_{ji} \]

\[ \times \left[ \frac{1}{\Delta^2_2(\text{ir}_2) \Delta^2_2(\text{i}s_2)} + \frac{1}{\Delta^2_2(\text{ir}_2) \Delta^2_2(\text{i}s_2)} \right] \right] 

\[ \times \sum_{j=1}^{3} \left( R_{11} R_{12} + (R_{11} + R_{12})(r_{21} - r_{11}) - (R_{11} + R_{12}) \left( \frac{\partial^-}{\partial \bar{s'}_1} \right) \right) \]

\[ \times \left( \frac{16}{\text{i}s_{12} - \bar{s'}_1} M_{22}(s'_1, s_1) - \frac{16}{\text{i}s_{12} - \bar{s'}_1} M_{12}(s'_1, s_1) \right) \]

\[ \times \left( \frac{1}{\Delta^2_2(\text{ir}_2) \Delta^2_2(\text{i}s_2)} + \frac{1}{\Delta^2_2(\text{ir}_2) \Delta^2_2(\text{i}s_2)} \right) \]

\[ \times \sum_{j=1}^{2} \left( R_{11} R_{12} + (R_{11} + R_{12})(r_{21} - r_{11}) - (R_{11} + R_{12}) \left( \frac{\partial^-}{\partial \bar{s'}_1} \right) \right) \]

\[ \times \exp(-r_{11}\text{tr} \, s'_1) \Phi^{(1)}_{3}(-is'_1, \bar{r}_1). \]  

(E8)
To further evaluate the expressions, we can now invoke a symmetry argument between the eigenvalues $r_{11}$ and $r_{21}$, respectively. Since the product $R_{11}R_{12}$ appears as a prefactor in front of the integral (E7), $R_{21}R_{22}$ must also appear as a prefactor in the final result. Thus, all terms in Eqs. (E7) and (E8) which do not contain $R_{21}R_{22}$ as a factor must yield zero. The remaining terms which are proportional to $R_{21}R_{22}$ can again be treated using formulas (4.7) and (5.9). However, we want to show explicitly that this line of arguing is correct and that the other terms indeed vanish. To this end, we need an additional identity to treat the operator product

$$
\sum_{j=1}^{2} \frac{\partial^{-}}{\partial s_{j1}} M_{ij}(s'_{1},s_{1}) \sum_{k=1}^{2} M_{2k}(s'_{1},s_{1}).
$$
(E9)

The required identity is given by the following formula: The same conditions as for formula (4.7) apply, furthermore we define

$$
L_{m}^{-}(s) L_{n}(s) = \sum_{i,j} \frac{1}{(is_{m2}-s_{i1})(is_{n2}-s_{j1})} \frac{\partial^{3}}{\partial s_{i1} \partial s_{j1}^{2}} + \frac{1}{2} \sum_{i,j} \frac{1}{(is_{m2}-s_{i1})(is_{n2}-s_{j1})}
\
\times \frac{\partial}{\partial s_{i1}} \sum_{k=1}^{k_{1}} \left( \frac{\partial}{\partial s_{j1}} - \frac{\partial}{\partial s_{k1}} \right).
$$
(E10)

Then we have

$$
L_{m}^{-}(s) \tilde{L}_{n}(s) \int_{s_{11}}^{s_{11}} \cdots \int_{s_{(k_{1}-1)1}}^{s_{11}} \mu_{B}(s',s) d[s'_{1}] f(s'_{1})
\int_{s_{11}}^{s_{11}} \cdots \int_{s_{(k_{1}-1)1}}^{s_{11}} \left[ \sum_{j=1}^{k_{1}} M_{m}^{-}(s'_{1},s_{1}) \frac{\partial^{-}}{\partial s_{j1}} M_{n}(s'_{1},s_{1}) f(s'_{1}) - \frac{1}{is_{n2}-is_{m2}}
\times \sum_{i=1}^{k_{1}-1} \left( \frac{1}{is_{m2}-s_{i1}} \frac{\partial^{-}}{\partial s_{i1}} M_{m}(s'_{1},s_{1}) - \frac{1}{is_{n2}-s_{i1}} \frac{\partial^{-}}{\partial s_{i1}} M_{m}(s'_{1},s_{1}) \right)
\times \sum_{j=1}^{k_{1}} M_{m}^{-}(s'_{1},s_{1}) \frac{\partial^{-}}{\partial s_{j1}} M_{n}(s'_{1},s_{1})
\times \frac{1}{(is_{m2}-s_{k1})(is_{n2}-s_{k1})(s'_{k1}-s_{k1})} \frac{\partial^{-}}{\partial s_{k1}}
\times \frac{1}{2} \sum_{k \neq j} \left( \frac{1}{(is_{m2}-s_{k1})(is_{n2}-s_{k1})(s'_{k1}-s_{k1})^{2}} \right)
\times \mu_{B}(s',s) d[s'_{1}] f(s'_{1}) \mu_{B}(s',s) d[s'_{1}].
$$
(E11)

The proof is similar to the one of formula (4.7). We notice that the arrow in Eq. (E10) is used slightly differently than previously. The operator $L_{m}^{-}(s)$ acts also on a part of $L_{n}(s)$. This is not consistent with the definition in Eq. (5.6). However, since this is obvious where it occurs, we still use the same arrow. We can now translate the left-hand side of Eq. (E8) into an expression in terms of $\Phi_{\lambda}^{(1)}(-is,r)$. After some further manipulations involving the identities in Eqs. (E6), (E1), and (E2) we arrive at
\[ \Phi_{44}(-is,r) = \hat{G}_{44} \exp(\text{tr} r_2 s_2 + r_{11} \text{tr} s_{11}) \prod_{i,j} R_{ji} \prod_{k=1}^{4} S_{k} \left[ \frac{1}{\Delta_2^2(\text{tr} r_2) \Delta_2^2(\text{tr} s_2)} + \frac{1}{\Delta_2^2(\text{tr} r_2) \Delta_2^2(\text{tr} s_2)} \right] \]
\[
\times \left(8R_{12}R_{22} - 4R_{22} \sum_{k=1}^{4} S_{k}^{-1} - 4L_2(s) \right) \left(8R_{11}R_{21} - 4R_{21} \sum_{k=1}^{4} S_{k}^{-1} - 4L_1(s) \right) \]
\[
- \frac{8}{\Delta_2^2(\text{tr} r_2) \Delta_2^2(\text{tr} s_2)} \left( \text{tr} r \sum_{i=1}^{4} S_{k}^{-1} \right) (L_1(s) - L_2(s)) \]
\[
- \frac{16}{\Delta_2^2(\text{tr} r_2) \Delta_2^2(\text{tr} s_2)} \sum_{k=1}^{4} \prod_{j=1}^{2} R_{2j} S_{k}^{-1} \right] \exp(-r_{11} \text{tr} s_{11}' \Phi_{4}(s_1', r_1)(i r_{12} \rightarrow i r_{22})). \]

(E12)

After rearranging terms this yields the result (5.18) for \( \Phi_{44}(-is,r) \).

30. N. Datta and H. Kunz, cond-mat/0006488.